Matrix Algebra Day 2 Notes

1.1 Notes - Lines & Linear Equations:

Linear equations: have form $a_1x_1 + a_2x_2 + \dots a_nx_n = b$ [ex: $3x_1 + 4x_2 + 7x_3 - 2x_4 = -19$]

A solution $(s_1, s_2, \dots s_n)$ is an ordered set of n-numbers (n-tuple) such that if $x_1 = s_1, x_2 = s_2, \dots x_n = s_n$ then [ex] is satisfied.

↳ A solution to [ex] is $(-2, 5, 1, 13)$, because $3(-2) + 4(5) - 7(1) - 2(13) = -19$

A solution set for a linear equation consists of all solutions to the equation.

↳ 2 variables = solution set is a line

↳ 3 variables = solution set is a plane

↳ 4+ variables = solution set is a hyperplane

A system of linear equations is a collection in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \dots a_{3n}x_n = b_3 \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_4 \end{cases}$$

a_{32} represents coefficient from 3rd eq, multiplied by x_2

Ex] Find all solutions to the system:

$$6x_1 - 10x_2 = 0$$
$$-3x_1 + 5x_2 = 8\} \cdot 2$$

$$\begin{aligned} 6x_1 - 10x_2 &= 0 \\ + \quad -6x_1 + 10x_2 &= 16 \\ \hline 0 &= 16 \end{aligned}$$

This tells us that there are no solutions to the system.

Ex] Find all solutions to the system:

$$(4x_1 + 10x_2 = 14) \cdot 3/2$$
$$-6x_1 - 15x_2 = -21$$

$$\Rightarrow \begin{aligned} 6x_1 + 15x_2 &= 21 \\ + \quad -6x_1 - 15x_2 &= -21 \\ \hline 0 &= 0 \end{aligned}$$

This tells us that the system is satisfied by any choices of x_1 & x_2, meaning that the relationship between x_1 & x_2 is the same in both equations. This means we can select either equation & solve for x_1 in terms of $x_2 \Rightarrow x_1 = \frac{7}{2} - \frac{5}{2}x_2$. For every choice of x_2 there will be a corresponding x_1 that satisfies the original system.

Therefore: $x_1 = \frac{7}{2} - \frac{5}{2}s_1$, $x_2 = s_1$ where s_1 is a free parameter and can be any real number. This is known as the general solution bc it gives all solutions to the system of equations

Triangular systems - use back substitution (starting at the bottom & going up)

Ex]

$$\begin{aligned} x_1 - 2x_2 - 5x_3 + 3x_4 &= 2 \\ x_2 + 3x_3 - 4x_4 &= 7 \\ x_3 + 2x_4 &= -4 \\ x_4 &= 5 \end{aligned}$$

$x_3 + 2(5) = -4$, $x_3 = -14$

$x_2 + 3(-14) - 4(5) = 7$, $x_2 = 69$

$x_1 - 2(69) - 5(-14) + 3(5) = 2$, $x_1 = 55$

Solution is $(55, 69, -14, 5)$

(All of these are leading variables)

1.1 Notes - Continued

A variable that appears as the first term in at least 1 equation = leading variable

Properties of Triangular Systems

(a) Every variable is the leading variable of exactly one equation

(b) There are the same number of equations as variables

(c) There is exactly one solution

Echelon form / Echelon system - organized in a descending stair step pattern from left to right, so that the indices of the leading variables are strictly increasing from top to bottom. ($c=0$ is below $c \neq 0$)

Properties of Echelon Systems

(a) Every variable is the leading variable of at most one equation

(b) There are 0 solutions, exactly one solutions, or infinitely many solutions

For a system in echelon form, any variable that is not a leading variable is called a free variable. x_3 below is a free variable.

Ex | $\left.\begin{aligned} 2x_1 - 4x_2 + 2x_3 + x_4 &= 11 \\ x_2 - x_3 + 2x_4 &= 5 \\ 3x_4 &= 9 \end{aligned}\right\}$ Back substitute, just like triangular systems, except set $x_3 = s_1$ (where s_1 is a free parameter), giving the general solution: $x_1 = 2 + s_1$, $x_2 = -1 + s_1$, $x_3 = s_1$, $x_4 = 3$

Total # of variables = # of free variables + # of leading variables